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MULTIPLE IMPACTS OF TWO CONCENTRIC HOLLOW CYLINDERS WITH ZERO CLEARANCE

X. C. YIN
Department of Applied Mechanics, Nanjing University of Science and Technology, Nanjing, 210014, P.R. China

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Abstract—The impact problem of two concentric, hollow, circular, elastic cylinders of same materials with zero clearance is studied, where the interior hollow cylinder is subjected to different uniformly distributed, exponentially decaying, interior pressures. The structural sizes and material parameters of the impacting system are unchanged. The influences of multiple collisions and multiple separations on interface impact pressures, dynamic radial displacements and stresses, are considered in the present investigation. The global histories of interface impact pressure have similar decay tendencies, and there exist three types of interface impact pressure branches. A so called 'group' phenomenon occurs in the histories of interface impact pressure, where several collisions take place closely together to form a cluster of interface impact pressure branches. Two neighboring cluster have an approximately constant time interval. The quasi-periodic growing of clusters means the impacting system is stable, and will result in quasi-periodic motions. A main frequency exists in the impacting system as well, which is different from the natural frequencies of two hollow cylinders. © 1997 Elsevier Science Ltd.

NOTATION

r	radial coordinate
t	time variable
A_1, B_1	inner, outer boundary surface of interior hollow cylinder
A_2, B_2	inner, outer boundary surface of outer hollow cylinder
a_1, b_1	inner, outer radius of interior hollow cylinder
a_2, b_2	inner, outer radius of outer hollow cylinder
$u(r, t)$	radial displacement field
$v(r, t)$	radial velocity field
$u_s(r, t)$	quasi-static radial displacement field
$\sigma_r(r, t)$	radial stress component
$\sigma_q(r, t)$	tangential stress component
ΔU	relative radial displacement between two surfaces A_2 and B_1
ΔV	relative radial velocity between two surfaces A_2 and B_1
N	total number of the actions including collisions and separations
t_N	the moment of beginning N-th action
t_N^-	immediately before t_N
$p_1(t)$	the history of interior pressure applied on inner boundary surface of the interior hollow cylinder
$p_2(t)$	the history of interface impact pressure between two surfaces A_2 and B_1
c	longitudinal wave velocity
λ, μ	Lame's constant
α	factor to show the rapidity of decay
σ_0	the amplitude of interior pressure
$H(t)$	Heaviside step function
J_n, Y_n	the Bessel functions of the first and second kind of order n
k_n, ω_n	wave number and natural frequency of order n for a single hollow cylinder
k_{nT}, ω_{nT}	wave number and natural frequency of order n for a single hollow cylinder with inner radius a_1 and outer radius b_2, which is called an assuming total hollow cylinder in the paper
$u_0(r)$	initial radial displacement distribution
$v_0(r)$	initial radial velocity distribution
$U(\cdots)$	the solution of $u(r, t)$ by a series of calculations from eqns (7)–(21)
T	$T = t \cdot c/(b_2 - a_1)$
T_0	$T_0 = (b_2 - a_1)/c$
σ_r	$\sigma_r = \sigma(r, t)/\sigma_0$
σ_θ	$\sigma_\theta = \sigma_q(r, t)/\sigma_0$
P_2	$P_2 = p_2(t)/\sigma_0$
U	$U = u(r, t)/(1e-12m)$

1. INTRODUCTION

One of the most important problems in elastodynamics is how the elastic waves propagate and interact at interfaces in layered media. This subject was usually constricted to the welded contact interfaces and infinite objects, and has appeared in several reference-books by Ewing *et al.* (1957), Brekhovskikh (1980), Achenbach (1973), Eringen and Suhubi (1975), and Miklowitz (1978), and a lot of papers reviewed by Pao (1983). Only the first separating or sliding process of interfaces was considered sometimes for infinite objects in Comninou *et al.* (1982), and Gong (1989). The impacting systems of finite structures, such as two rods, a mass on a rod or a beam, two spheres or a sphere on a plate, were studied by one-dimensional theory and other approximate theories in Goldsmith (1960) and Johnson (1987), but only the initial period of contact was investigated.

In recent years, there has been a rapid development in the study of collisions caused by an elastic component colliding with rigid obstacles or two elastic components colliding with each other. To simplify the analyses of the impact process with multiple collisions and multiple separations, some vibrating models were presented for these impacting systems. Although these impacting systems were modeled as an impact pair in Han *et al.* (1995), an acceleration damper in Grubin (1956), an impact oscillator in Budd *et al.* (1995), and an impact damper in Bapat and Sankar (1985), the impact oscillations were still complex and highly non-linear. There are different time intervals existing between a collision and a separation, but the vibrating models described a separation immediately following a collision, so that many natural properties are lost.

This paper investigates the radial impact of two perfectly concentric, hollow, circular, elastic cylinders. Two hollow cylinders with zero clearance have the same materials. The inner boundary surface of interior hollow cylinder is subjected to a uniformly distributed, exponentially decaying, axially symmetric, interior pressure. The investigation is concentrated on interface impact pressures, dynamic radial displacements and stresses in axially symmetric plain strain state, which are calculated using the solutions by eigenfunction expansions for different applied interior pressures. Multiple collisions and multiple separations of two hollow cylinders are considered. The calculating results show that two hollow cylinders will keep contact different periods of time during collisions, which is much different from the description in the vibrating models. From the calculating results, some properties of interface impact pressure, the main frequency of the impacting system, and the variations in dynamic radial displacement and stresses, are obtained in the present paper as well.

2. COLLISION AND SEPARATION CONDITIONS

We assume two hollow cylinders have perfectly smooth, concentric circular surfaces, same materials and zero clearance. The unchanged structural sizes are $a_1 = 1$ m, $a_2 = b_1 = 3$ m, $b_2 = 6$ m, $c = 5000$ m/s, $\lambda = \mu = 80$ GPa. Two hollow cylinders are initially at rest and inner boundary surface of interior hollow cylinder is subjected to a uniformly distributed, exponentially decaying, axially symmetric, interior pressure

$$p_1(t) = \sigma_0 \cdot e^{-\alpha t} H(t) \tag{1}$$

where $\sigma_0 = 1$ Pa is selected.

The initial collision occurs as pressure waves march to the surface A_2 at time $(a_2 - a_1)/c$. Then the first branch of interface impact pressure is generated. Two hollow cylinders keep contact until the interface impact pressure become zero, then the first separation occurs immediately.

When two hollow cylinders are separated, ΔU is larger than zero. The next collision occurs at the time as ΔU is equal to zero.

When two hollow cylinders are in contact, the interface impact pressure is larger than zero. The next separation occurs as $p_2(t)$ is equal to zero.

The collision and separation conditions are, respectively,

$$t_1 = (b_1 - a_1)/c \quad \text{beginning of the first collision} \tag{2}$$

$$p_2(t) = 0 \quad \frac{\mathrm{d}p_2(t)}{\mathrm{d}t} \leqslant 0 \quad \text{beginning of a separation} \tag{3}$$

$$\Delta U = 0 \quad \Delta V \geqslant 0 \quad \text{beginning of a collision.} \tag{4}$$

Two limited states might exist as

$$\Delta U = 0 \quad \Delta V = 0 \tag{5}$$

$$p_2(t) = 0 \quad \frac{\mathrm{d}p_2(t)}{\mathrm{d}t} = 0. \tag{6}$$

These states are referred to as 'graze' in Budd *et al.* (1995).

3. SOLUTIONS OF MULTIPLE COLLISIONS AND MULTIPLE SEPARATIONS

The solutions by eigenfunction expansions can be applied to a single hollow circular elastic cylinder with inner radius a and outer radius b, where inner boundary surface is subjected to a uniformly distributed, axially symmetric, interior pressure $p(t)$. The expansion theorem states that the general solution (equation (5.17.12) in Eringen and Suhubi, 1975) can be expressed as

$$u(r,t) = u_s(r,t) + \sum_n U_n(r) q_n(t). \tag{7}$$

It should be noted that equation (5.17.12) is correct, but equation (5.17.19) is wrong in Eringen and Suhubi (1975), which was discussed in Gong and Wang (1991) and Wang and Gong (1991). $u_s(r, t)$ satisfies inhomogeneous boundary conditions, and eigenfunctions $U_n(r)$ satisfy homogeneous boundary conditions

$$u_s(r,t) = \varphi_1(r,a,b) p(t) \tag{8}$$

$$\varphi_1(r,a,b) = \frac{a^2}{2(\lambda+\mu)(b^2-a^2)} r + \frac{a^2 b^2}{2\mu(b^2-a^2)} \frac{1}{r} \tag{9}$$

$$U_n(r) = A_n \varepsilon_1(k_n r) \tag{10}$$

$$\varepsilon_1(k_n r) = M_2(k_n, a) J_1(k_n r) - M_1(k_n, a) Y_1(k_n r) \tag{11}$$

$$M_1(k_n, r) = k_n J_1'(k_n r) + \frac{\lambda}{\lambda+2\mu} \frac{1}{r} J_1(k_n r) \tag{12}$$

$$M_2(k_n, r) = k_n Y_1'(k_n r) + \frac{\lambda}{\lambda+2\mu} \frac{1}{r} Y_1(k_n r) \tag{13}$$

k_n, ω_n are calculated from the frequency equation in Gazis (1958).

$$M_1(k,a) M_2(k,b) - M_1(k,b) M_2(k,a) = 0 \tag{14}$$

and

$$\omega_n = k_n \cdot c. \tag{15}$$

The factors A_n are determined by the normalization condition in Eringen and Suhubi (1975).

$$2\pi \int_a^b U_n^2(r) r \, \mathrm{d}r = 1. \tag{16}$$

In order to consider multiple collisions and multiple separations, we improve the solution of unknown time functions $q_n(t)$ in Eringen and Suhubi (1975). By introducing a time variable t^* denoting the beginning of a new collision or a new separation $q_n(t)$ is

$$q_n(t) = q_n(0) \cos \omega_n (t - t^*) + \frac{1}{\omega_n} \dot{q}_n(0) \sin \omega_n (t - t^*) + \frac{1}{\omega_n} \int_0^{t-t^*} \ddot{Q}_n(t^* + \tau) \sin \omega_n (t - t^* - \tau) \, \mathrm{d}\tau \tag{17}$$

$$Q_n(t) = -2\pi \int_a^b u_s(r, t) U_n(r) r \, \mathrm{d}r \tag{18}$$

$$q_n(0) = 2\pi \int_a^b u_0(r) U_n(r) r \, \mathrm{d}r + Q_n(t^*) \tag{19}$$

$$\dot{q}_n(0) = 2\pi \int_a^b v_0(r) U_n(r) r \, \mathrm{d}r + \dot{Q}_n(t^*) \tag{20}$$

$u_0(r)$ and $v_0(r)$ are the distributions immediately before time t^*. If t^* is zero, eqns (17)–(20) will give same results as equations (5.17.15) and (5.17.16) in Eringen and Suhubi (1975). From eqns (7)–(20), $u(r, t)$ and $v(r, t)$ can be expressed as the following functions

$$u(r, t) = U(r, t, t^*, c, \lambda, \mu, a, b, u_0(r), v_0(r), p(t))$$
$$v(r, t) = \frac{\partial U}{\partial t}(r, t, t^*, c, \lambda, \mu, a, b, u_0(r), v_0(r), p(t)). \tag{21}$$

For the impacting system in this paper, the solutions are

$$u(r, t) = \begin{cases} U(r, t, 0, c, \lambda, \mu, a_1, b_1, 0, 0, p_1(t)) & a_1 \leqslant r \leqslant b_1 \\ 0 & a_2 \leqslant r \leqslant b_2 \end{cases}$$
$$t_1 = (a_2 - a_1)/c \quad 0 \leqslant t \leqslant t_1^- \tag{22}$$

$$u(r, t) = U(r, t, t_1, c, \lambda, \mu, a_1, b_2, u(r, t_1^-), v(r, t_1^{-1}), p_1(t)) \quad a_1 \leqslant r \leqslant b_2$$
$$t_2 \text{ is determined by eqn (3)} \qquad t_1 \leqslant t \leqslant t_2^- \tag{23}$$

$$u(r, t) = \begin{cases} U(r, t, t_2, c, \lambda, \mu, a_1, b_1, u(r, t_2^-), v(r, t_2^-), p_1(t)) & a_1 \leqslant r \leqslant b_1 \\ U(r, t, t_2, c, \lambda, \mu, a_2, b_2, u(r, t_2^-), v(r, t_2^-), 0) & a_2 \leqslant r \leqslant b_2 \end{cases}$$
$$t_3 \text{ is determined by eqn (4)} \qquad t_2 \leqslant t \leqslant t_3^- \tag{24}$$

... ...

$$u(r, t) = \begin{cases} U(r, t, t_{2m}, c, \lambda, \mu, a_1, b_1, u(r, t_{2m}^-), v(r, t_{2m}^-), p_1(t)) & a_1 \leqslant r \leqslant b_1 \\ U(r, t, t_{2m}, c, \lambda, \mu, a_2, b_2, u(t, t_{2m}^-), v(r, t_{2m}^-), 0) & a_2 \leqslant r \leqslant b_2 \end{cases}$$
$$t_{2m+1} \text{ is determined by eqn (4)} \qquad t_{2m} \leqslant t \leqslant t_{2m+1}^- \tag{25}$$

$$u(r,t) = U(r,t,t_{2m+1},c,\lambda,\mu,a_1,b_2,u(r,t_{2m+1}^-),v(r,t_{2m+1}^-),p_1(t)) \quad a_1 \leqslant r \leqslant b_2$$

$$t_{2m+2} \text{ is determined by eqn (3)} \qquad t_{2m+1} \leqslant t \leqslant t_{2m+2}^- \tag{26}$$

The frequencies of interior and outer hollow cylinders are determined by eqn (14). The frequencies ω_{nT} of the assuming total hollow cylinder can also be determined by eqn (14). However, when two hollow cylinders are in contact, the frequencies required in eqn (26) must be determined by boundary conditions and additional continuity conditions of radial stress and radial displacement at interfaces, thus the frequency equation becomes

$$\begin{vmatrix} M_1(k,a_1) & M_2(k,a_1) & 0 & 0 \\ 0 & 0 & M_1(k,b_2) & M_2(k,b_2) \\ J_1(kb_1) & Y_1(kb_1) & -J_1(ka_2) & -Y_1(ka_2) \\ M_1(k,b_1) & M_2(k,b_1) & -M_1(k,a_2) & -M_2(k,a_2) \end{vmatrix} = 0 \tag{27}$$

which reduces to

$$\begin{vmatrix} M_1(k,a_1) & M_2(k,a_1) \\ M_1(k,b_2) & M_2(k,b_2) \end{vmatrix} \begin{vmatrix} J_1(kb_1) & Y_1(kb_1) \\ J_1'(kb_1) & Y_1'(kb_1) \end{vmatrix} = 0. \tag{28}$$

If we define a function

$$f(x) = J_1(x)Y_1'(x) - J_1'(x)Y_1(x) \tag{29}$$

then the derivative of $f(x)$ is

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -\frac{1}{x}f(x). \tag{30}$$

The general solution for $f(x)$ is

$$f(x) = c_1\frac{1}{x} \tag{31}$$

c_1 is given from a known result

$$x \to 0 \quad f(x) \to \frac{2}{\pi x} \tag{32}$$

so that

$$f(kb_1) = \frac{2}{\pi kb_1} > 0 \quad \text{as } kb_1 > 0. \tag{33}$$

It is clear that eqn (28) gives same results as eqn (14). In other words, when two hollow cylinders are in contact, the frequencies required in eqn (26) can be replaced directly by the natural frequencies of the assuming total hollow cylinder.

For a given $u(r,t)$, the stress components are

$$\sigma_r = (\lambda+2\mu)\frac{\partial u(r,t)}{\partial t} + \lambda\frac{u(r,t)}{r} \tag{34}$$

$$\sigma_q = \lambda\frac{\partial u(r,t)}{\partial t} + (\lambda+2\mu)\frac{u(r,t)}{r}. \tag{35}$$

4. INTERFACE IMPACT PRESSURE

Different time intervals exist between two neighboring collisions, which are listed in Table 1 for $\alpha = 50$. Some collisions occur closely, others may wait a longer period of time before finishing a separation. Two hollow cylinders keep contact the largest time interval during the initial collision.

The interface impact pressure $p_2(t)$ is equal to the negative radial stress at $r = b_1$. The global histories of $p_2(t)$ for different values of α are shown in Fig. 1. All values of N exceed one thousand. Two properties exist. The first is that all global histories have similar decay tendencies for different interior pressures, even for a jumping interior pressure as $\alpha = 0$ and a very short shock interior pressure as $\alpha = 5000$. The second is 'group' phenomenon which can be seen in Fig. 1, where several interface impact pressure branches collect closely together to form a cluster. These clusters grow quasi-periodically. The numbers of main clusters for different values of α are shown in Fig. 2. Some secondary clusters separate from main clusters, which become more clear as $\alpha = 5000$ in Fig. 1(g).

Above properties mean that the impacting system is dominated mainly by structure and material parameters. The impacting system is stable under given conditions and parameters. The 'group' phenomenon also implies a natural property that multi-body impacts have intermittence.

Each branch of interface impact pressure rises sharply from zero and then decays to zero. There are three types of the branches, smooth decay shape, single sharp peak shape, and multi-peak shape, which can be seen in Fig. 3. Smooth decay shape is formed in the initial collision, and equal to the shape of negative radial stress of the assuming total hollow cylinder at $r = b_1$ in the same time interval. It jumps initially, then decays with the widest time interval. Single sharp peak shapes exist in every plot in Fig. 3. Sometimes its peak value is very small. It may be foreseen the 'graze' might generate with a zero peak value, but it is difficult to predict by normal calculating methods. Multi-peak shapes also occur in

Table 1. The beginning time of a collision and a separation for $\alpha = 50$

N	Colliding time (T_0)	Separating time (T_0)	Time interval between two collisions (T_0)
1	0.400		
2		1.600	
3	2.572		0.972
4		2.800	
5	3.320		0.520
6		4.000	
7	4.388		0.388
8		4.925	
9	5.120		0.195
10		5.188	
11	5.513		0.325
12		5.727	
13	6.694		0.967
14		6.720	
15	7.714		0.994
16		7.912	
17	8.050		0.136
18		8.174	
19	8.228		0.054
20		8.293	
21	8.358		0.065

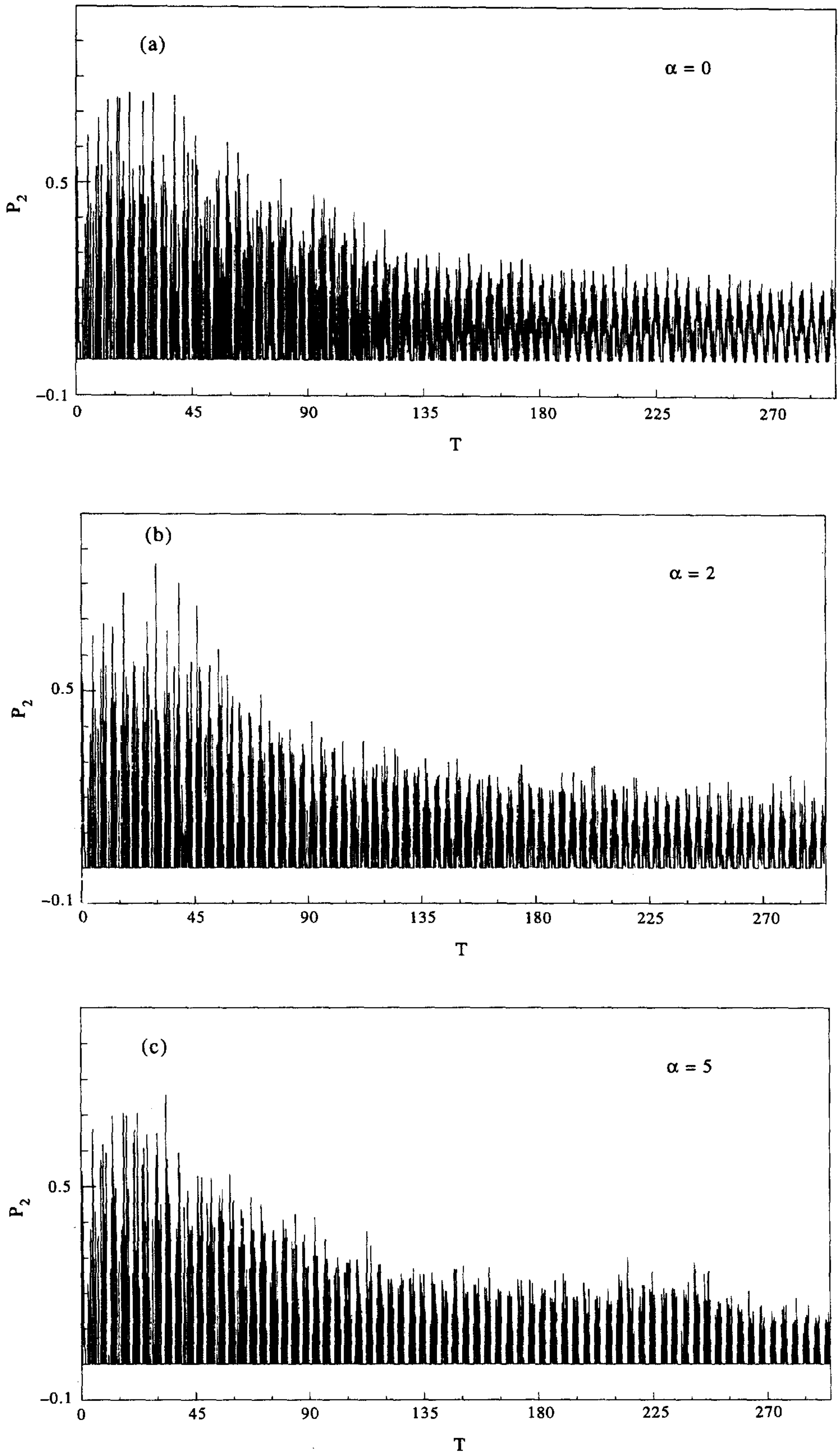


Fig. 1. Global histories of interface impact pressure: (a) $\alpha = 0$; (b) $\alpha = 2$; (c) $\alpha = 5$; (d) $\alpha = 50$; (e) $\alpha = 300$; (f) $\alpha = 500$; (g) $\alpha = 5000$.

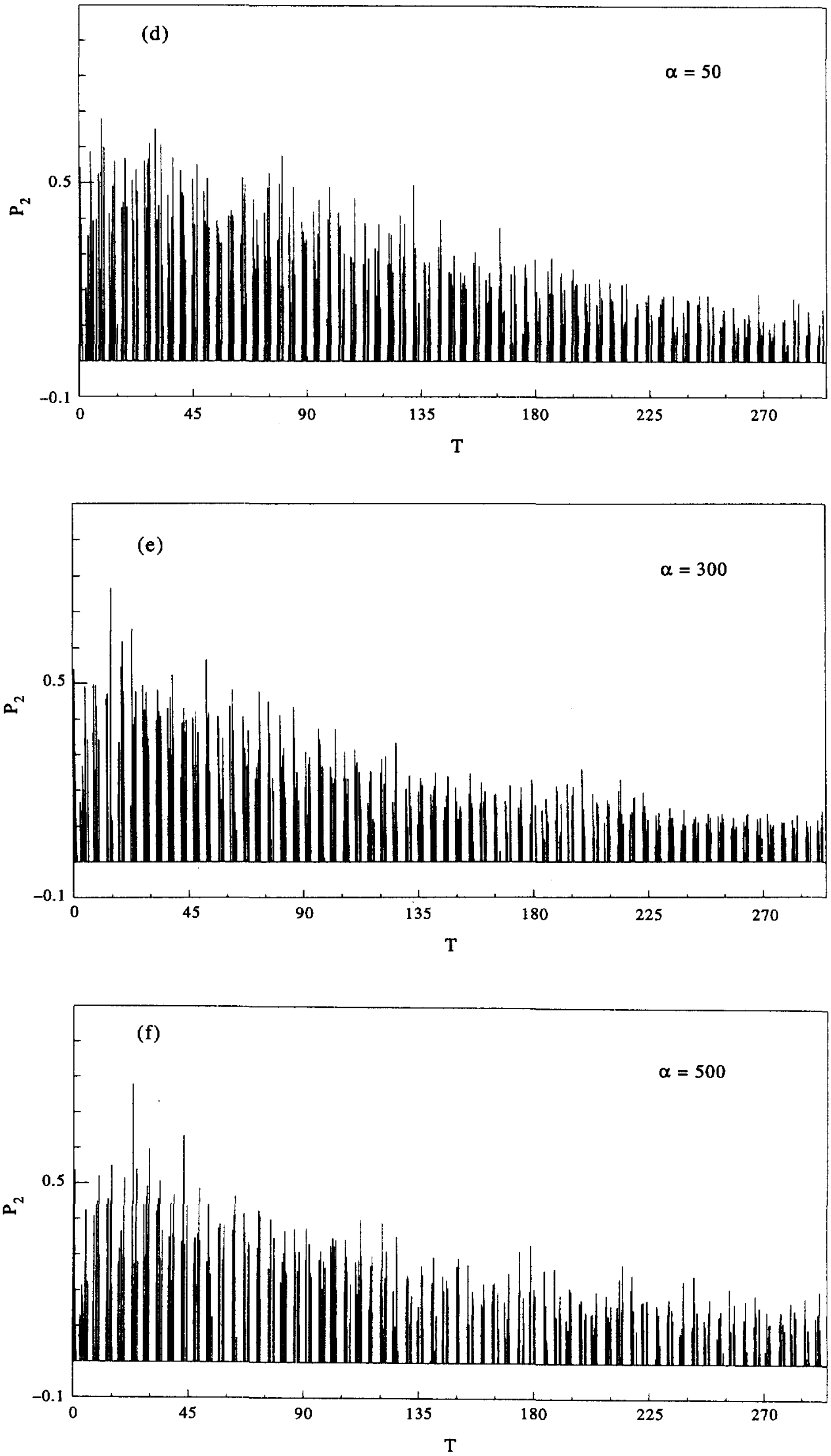


Fig. 1—*Continued*

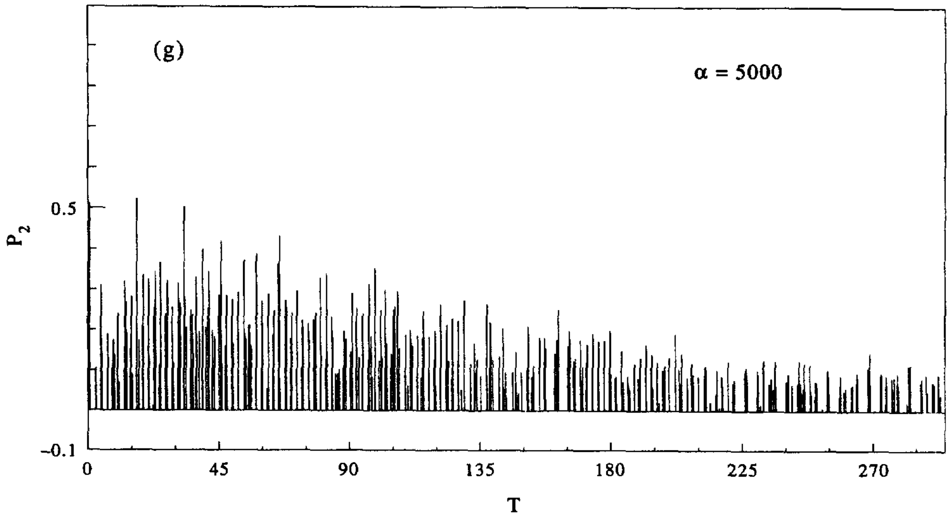


Fig. 1—*Continued*

every plot in Fig. 3. The numbers of peaks are over two. A special form is shown in Fig. 3(b), where a cluster may consist of several short branches and a wider branch with many peaks.

Multiple collisions and multiple separations strongly influence the propagation of stress waves. The distributions of radial stress for $\alpha = 50$ at $T = 3.8$, 12 and 102.2889 are shown in Fig. 4. Their values of N are 3, 39 and 675. It can be seen that the oscillations are mainly made by interface impact pressure waves by comparing the distributions in Fig. 4(a) and 4(d).

5. DYNAMICAL RADIAL DISPLACEMENT AND STRESSES

Radial displacements of outer boundary surface of outer hollow cylinder can be easily detected, and can provide some useful information of the impacting system, such as response frequencies. Figure 5 shows the histories of radial displacement of outer boundary surface of outer hollow cylinder for different α values. Radial displacements oscillate periodically, which indicate the impacting system is stable and may exist a main frequency. Usually, a single hollow cylinder subjected to the interior pressure $p_1(t)$ will oscillate on its natural

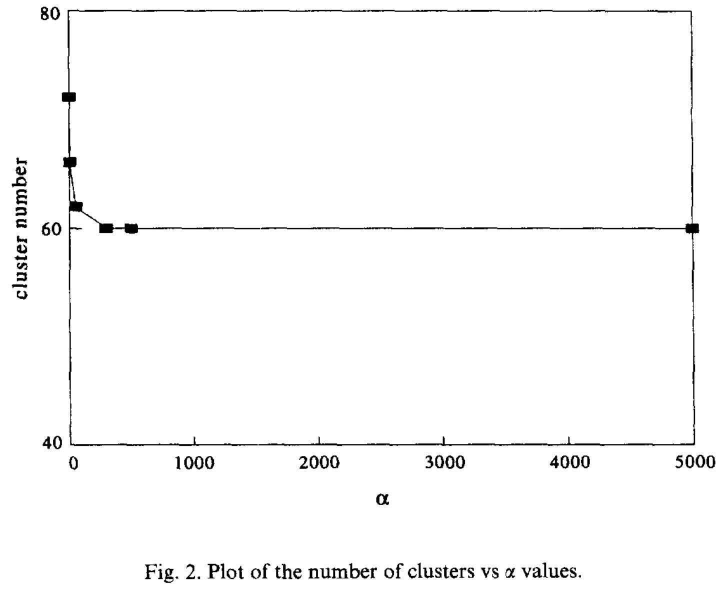


Fig. 2. Plot of the number of clusters vs α values.

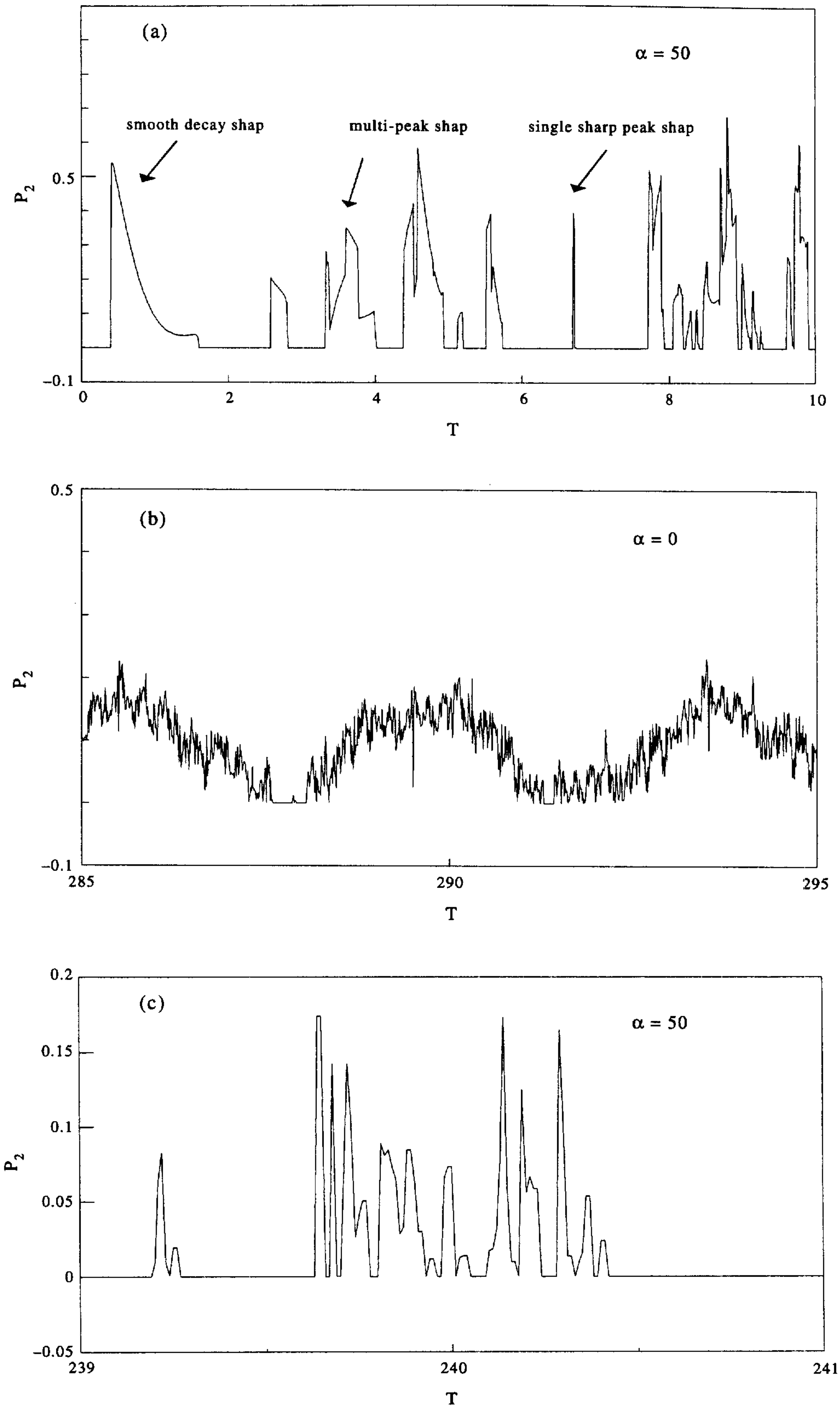


Fig. 3. Amplifications of clusters: (a) $\alpha = 50$; $T = 0$–10; (b) $\alpha = 0$; $T = 285$–295; (c) $\alpha = 50$; $T = 239$–241; (d) $\alpha = 50$; $T = 270$–290.

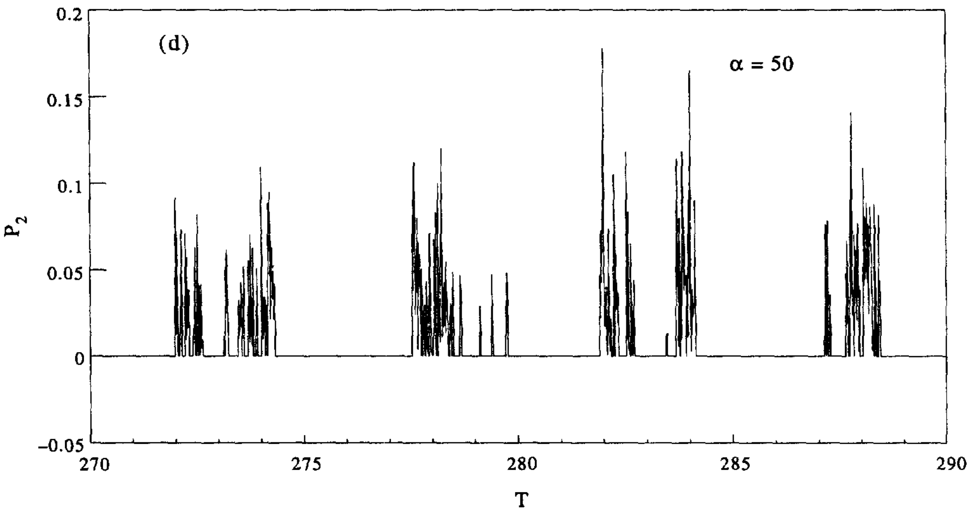
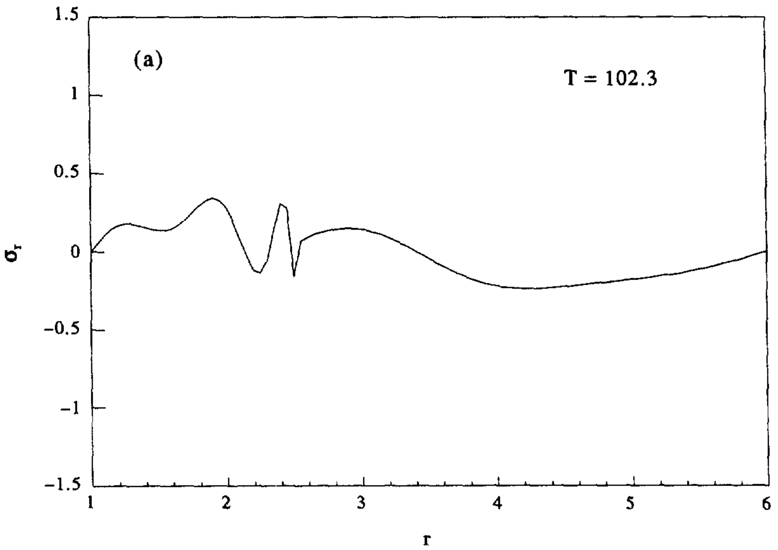


Fig. 3—*Continued*



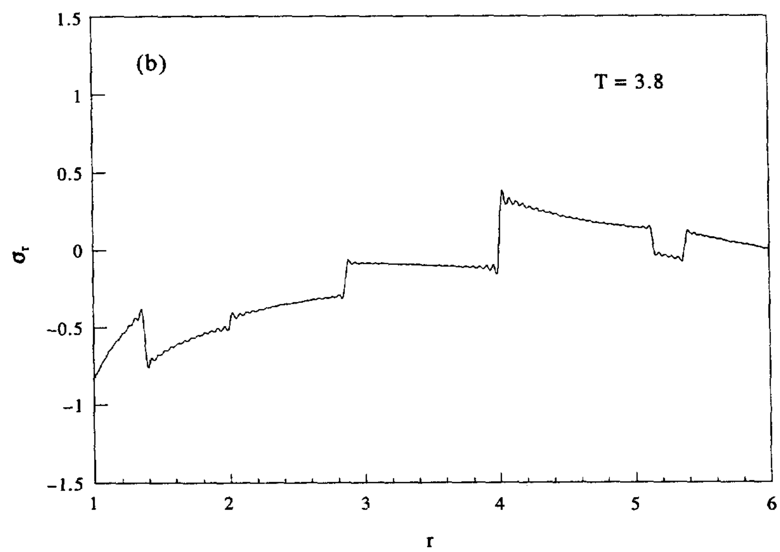


Fig. 4. Distributions of radial stress for $\alpha = 50$: for the assuming total hollow cylinder, (a) $T = 102.3$; for the impacting system, (b) $T = 3.8$; (c) $T = 12$; (d) $T = 102.3$.

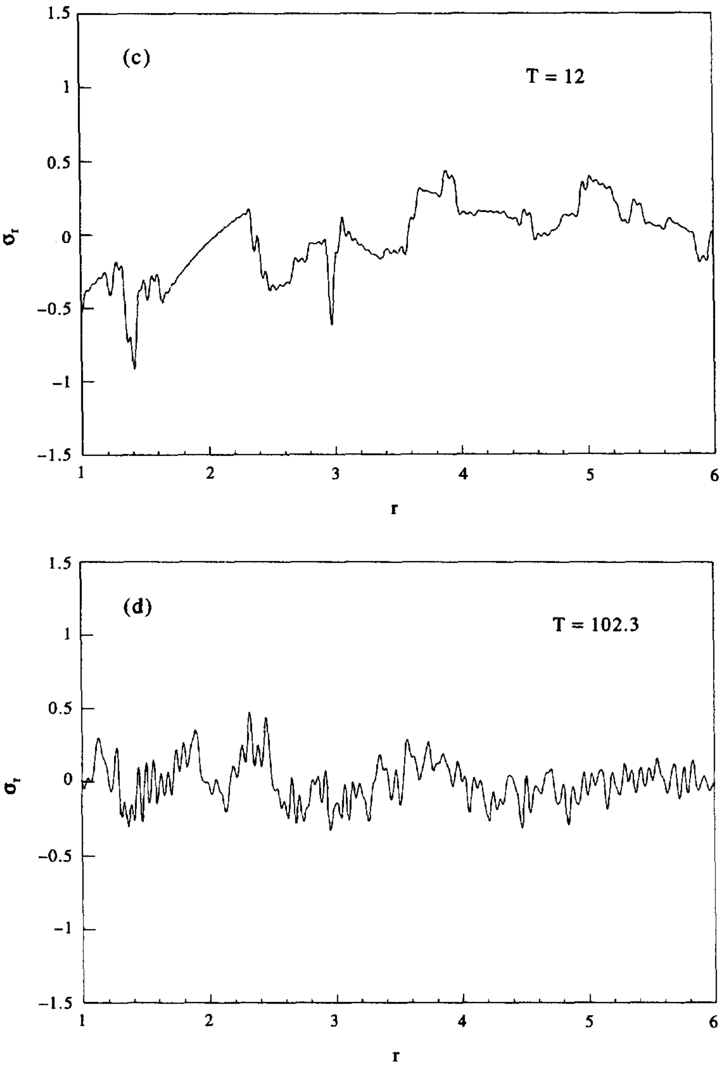
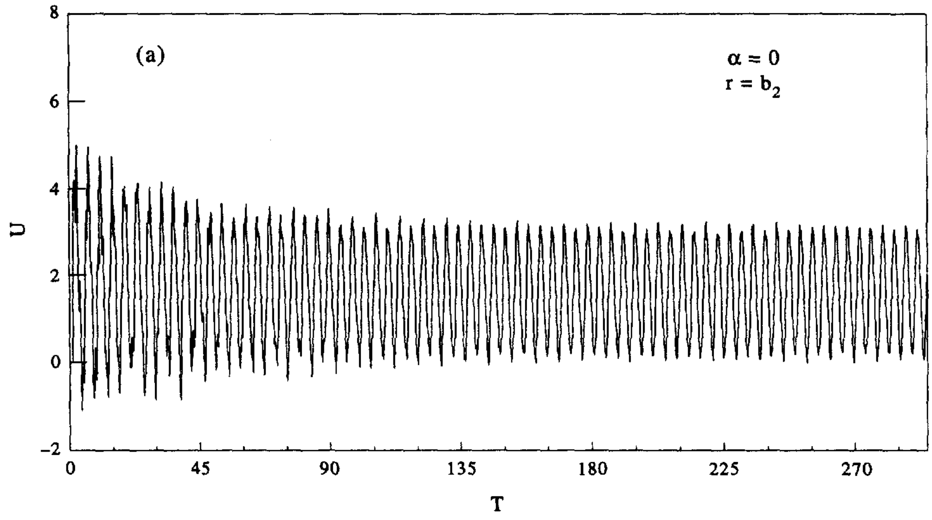


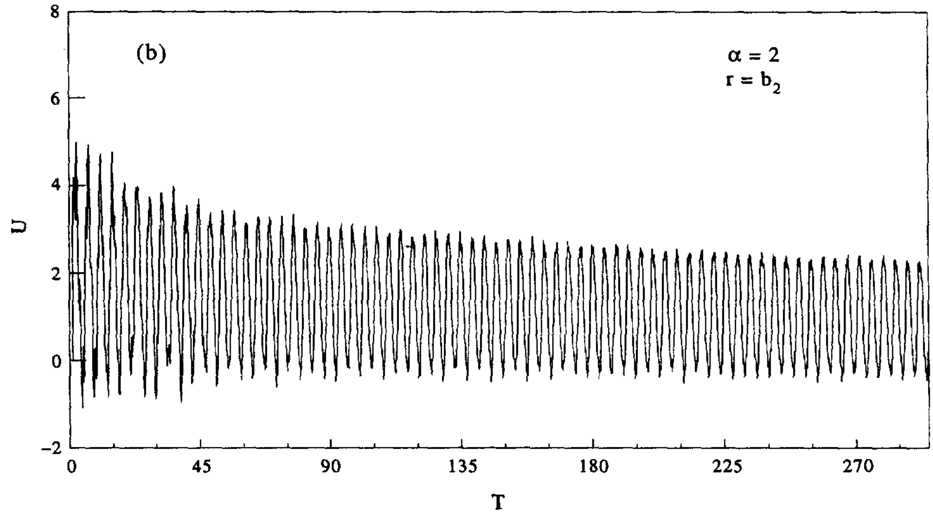
Fig. 4—*Continued*

frequencies, but in the impacting system, the response frequencies obtained by FFT method from the history of radial displacement, are different from the natural frequencies of interior, outer hollow cylinders and the assuming total hollow cylinder. The basic frequency is 1092.8 l/s for outer hollow cylinder, and 1559.9 l/s for the assuming total hollow cylinder. The main frequencies of the impacting system with different α values are listed in Table 2. The main frequencies are between these two basic frequencies and vary with values of α, so the impacting system is influenced by interior pressure $p_1(t)$. The change of frequency is an important information given by multiple collisions and multiple separations, which may have wide applications.

Figure 6 shows the histories of radial stress and tangential stress at different surface for $\alpha = 50$. The history of radial stress strictly coincident with interior pressure at $r = a_1$ and vanishing at $r = b_2$ shows that the boundary conditions are satisfied. The histories of radial stress at $r = a_2$ and $r = b_1$ being identical show that the stress continuity condition is satisfied during the periods of contact. The radial displacement continuity condition is satisfied as well.

Multiple impacts change the values of stresses and displacements, and generate rapidly secondary oscillations more and more frequently. The secondary oscillations will become small at $r = b_2$. It is convenient to select this surface for calculating main frequencies.





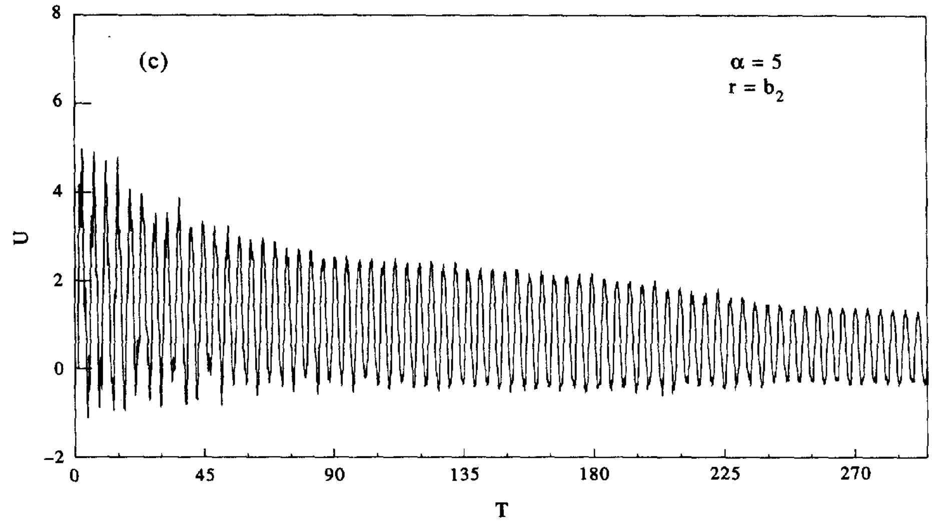


Fig. 5. Histories of radial displacements of outer boundary surface of outer hollow cylinder for different α values: (a) $\alpha = 0$; (b) $\alpha = 2$; (c) $\alpha = 5$; (d) $\alpha = 50$; (e) $\alpha = 300$; (f) $\alpha = 500$; (g) $\alpha = 5000$.

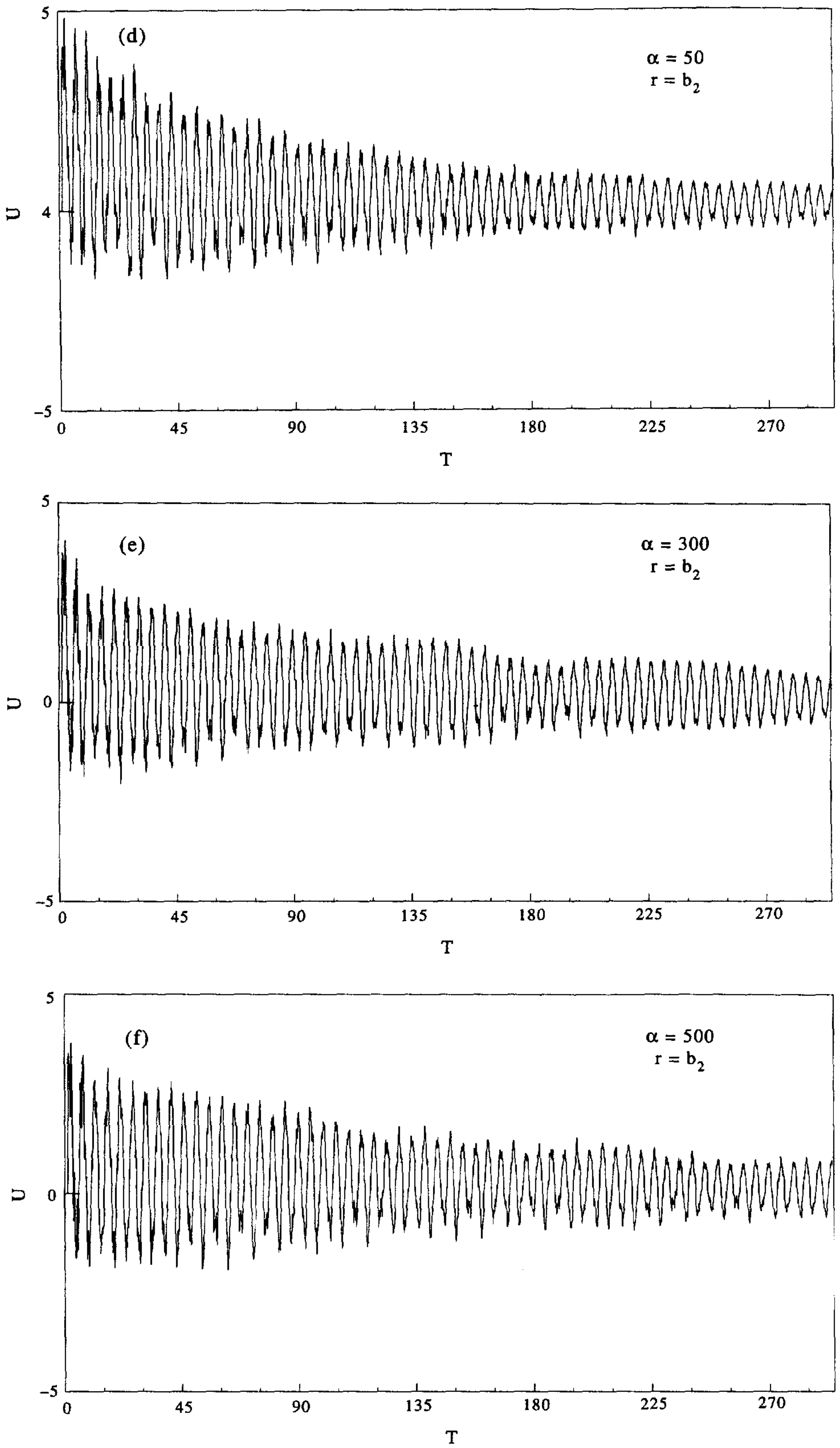


Fig. 5—*Continued*

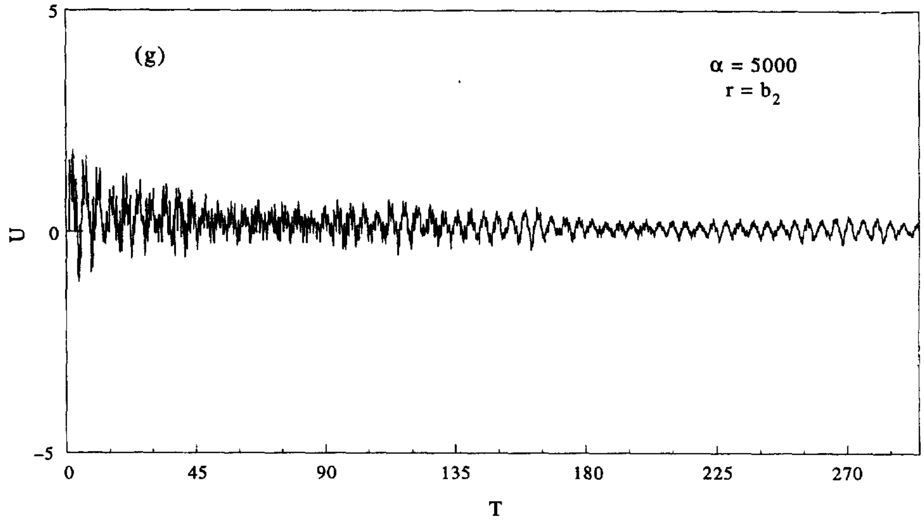


Fig. 5—*Continued*

Table 2. Main frequencies of the impacting system

α	0	2	5	50	300	500	5000
Frequency (1/s)	1548	1521	1449	1293	1287	1284	1344

6. CONCLUSIONS

Multiple collisions are multiple separations are first studied in elastodynamics for the impacting system in the present paper. More than one thousand times of collisions and separations are generated at considering time interval. Following conclusions are obtained:

1. The global histories of interface impact pressure have similar decay tendencies, even for the jumping interior pressure and the very short shock interior pressure.
2. 'Group' phenomenon occurs in the histories of interface impact pressure. Several interface impact pressure branches collect closely together to form a main cluster. Secondary clusters separated from main clusters are more clearly as $\alpha = 5000$.
3. The periodic growing of clusters tell us the impacting system in this paper will be stable.
4. Multiple impacts change the values of stresses and displacements, and caused more rapidly secondary oscillations.
5. The periodic growing of cluster results in the periodicity of radial displacements at $r = b_2$.
6. A main frequency exists in the impacting system, and varies with values of α, so that the influence of applied load on the impacting system must be considered.
7. The impacting system is stable for given structural sizes and material parameters in this paper, but we must emphasize that the impacting system will be unstable with other structural sizes.
8. The shapes of interface impact pressure have three types, smooth decay shape, single sharp peak shape and multi-peak shape.

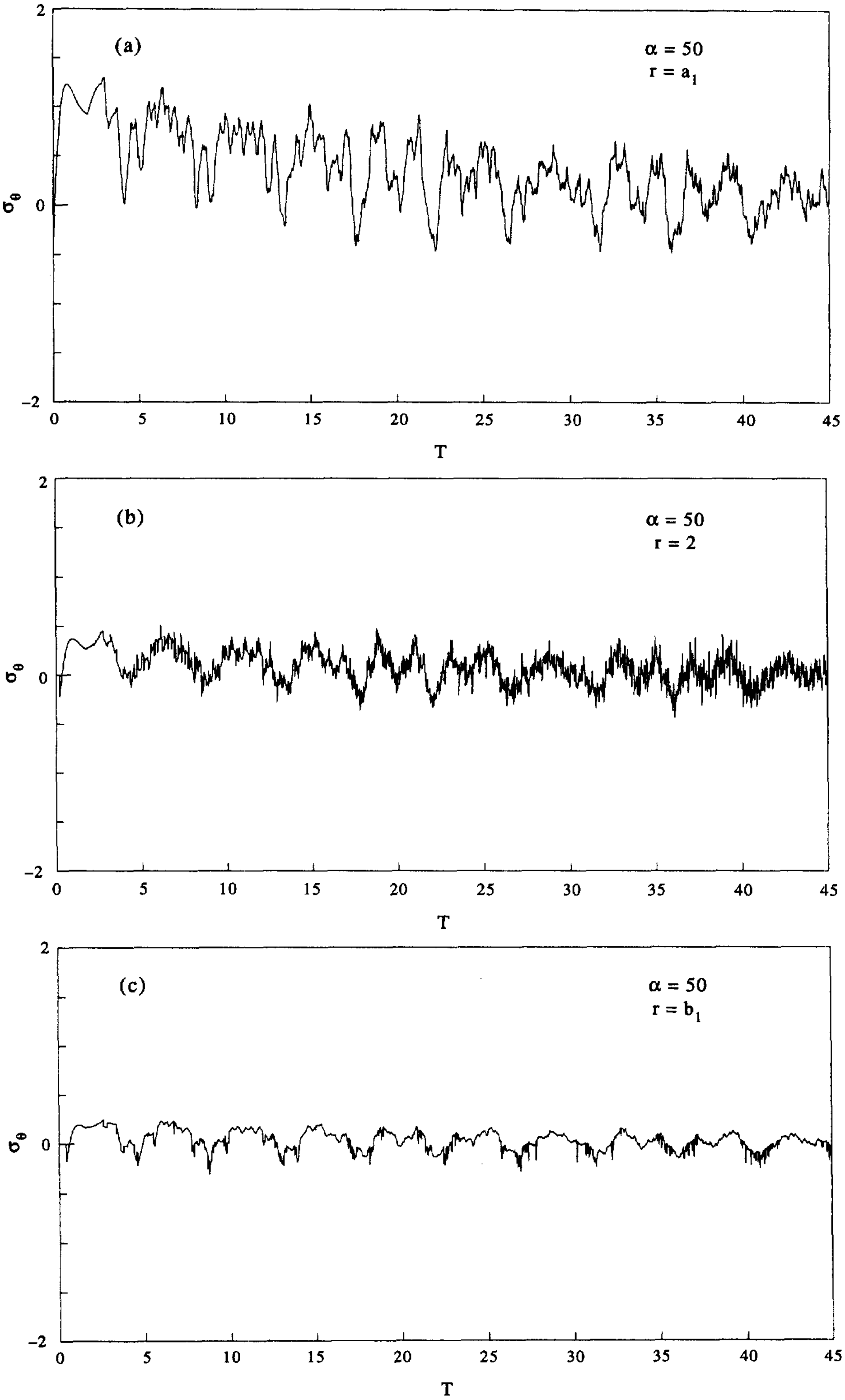


Fig. 6. Histories of stress for $\alpha = 50$: for tangential stress, (a) $r = a_1$; (b) $r = 2$; (c) $r = b_1$; (d) $r = a_2$; (e) $r = 4$; (f) $r = b_2$; for radial stress, (g) $r = a_1$; (h) $r = 2$; (i) $r = b_1$; (j) $r = a_2$; (k) $r = 4$; (l) $r = b_2$.

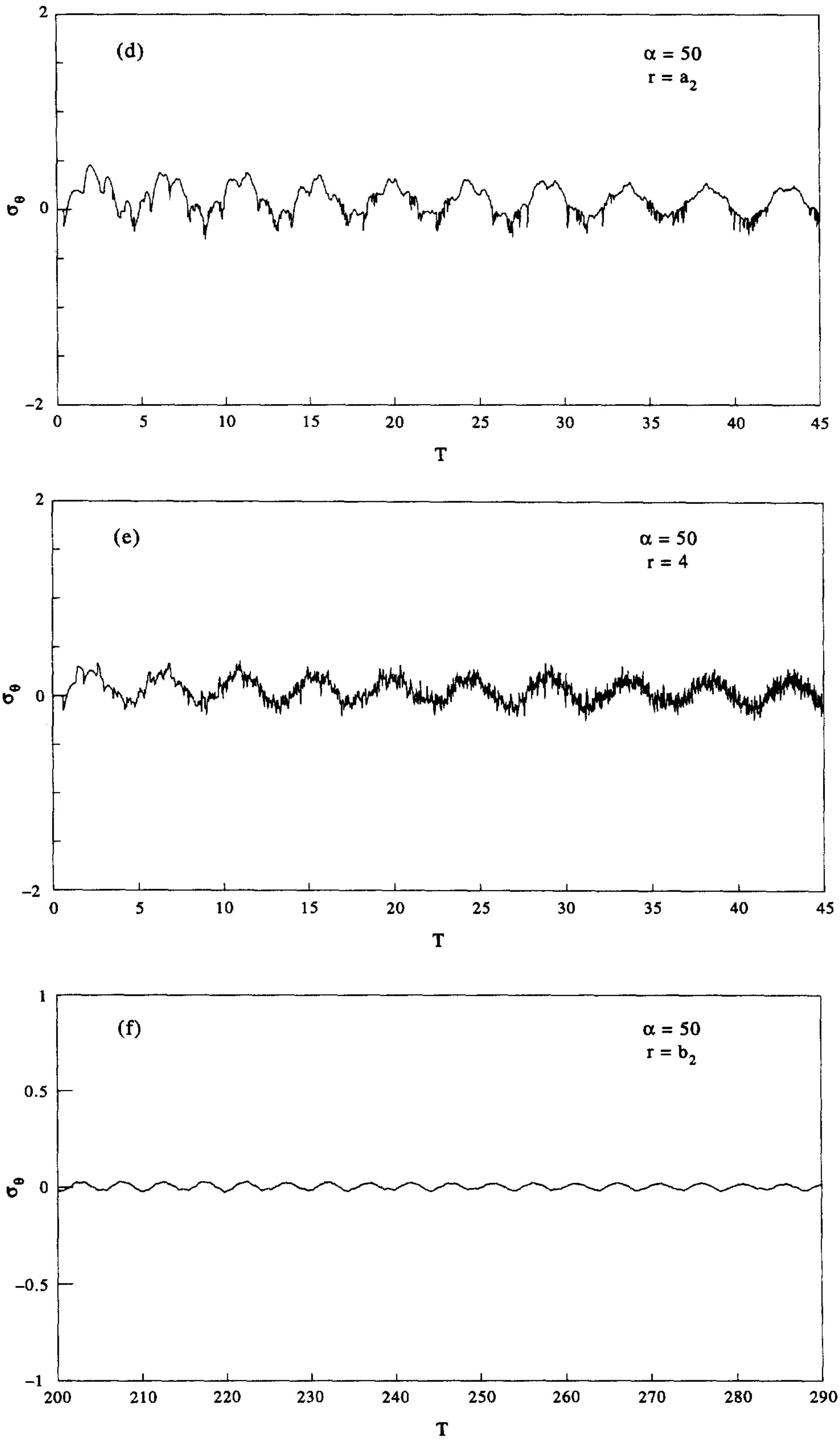


Fig. 6—*Continued*

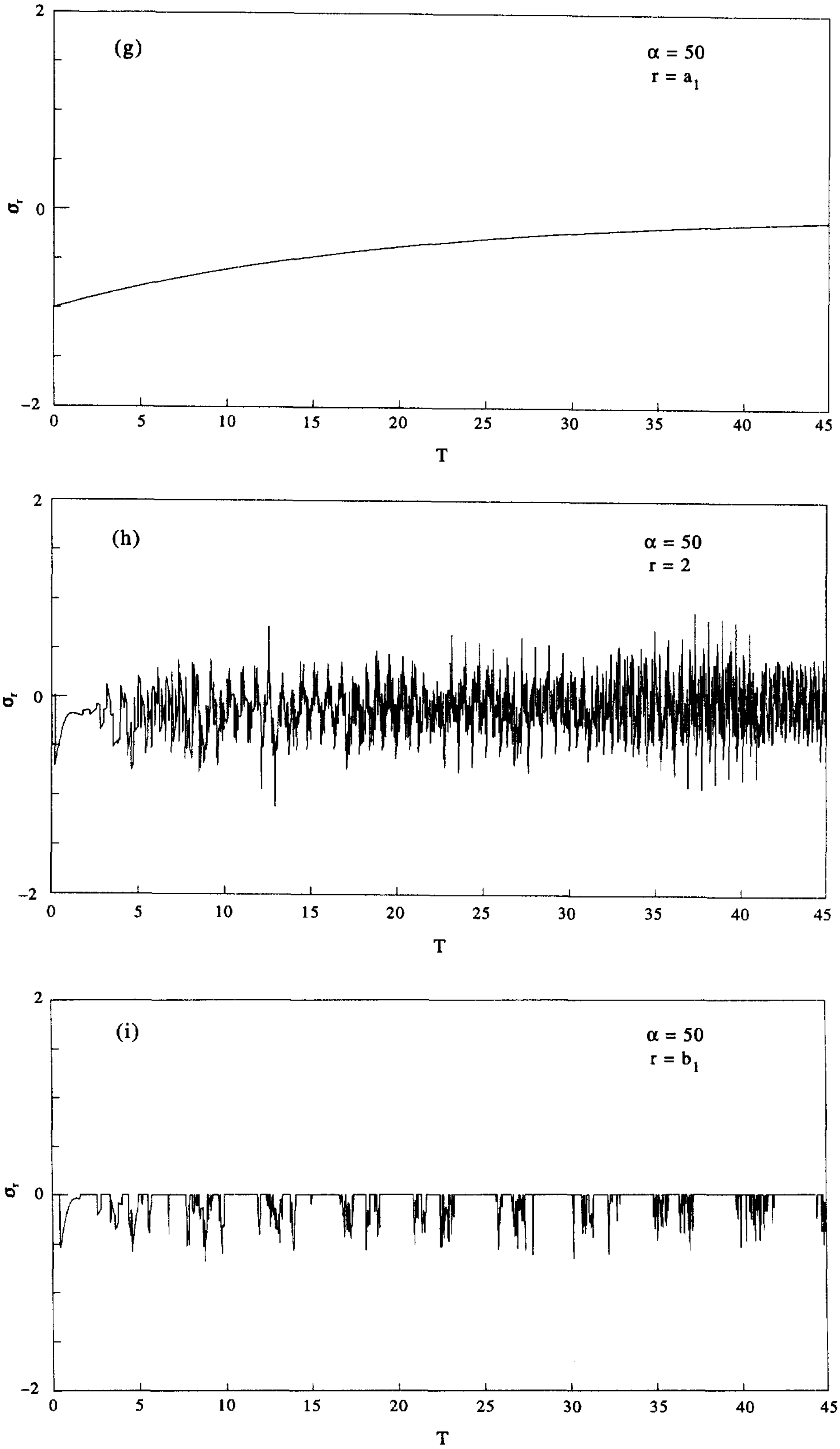
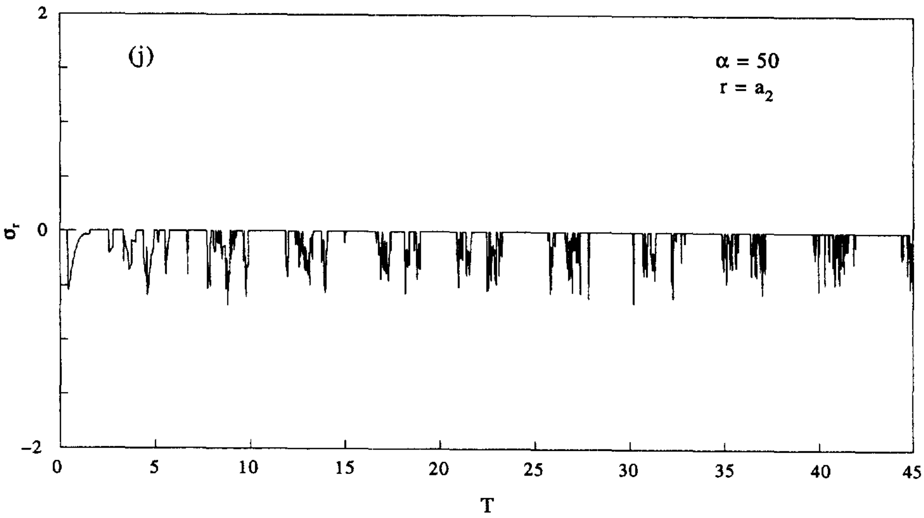
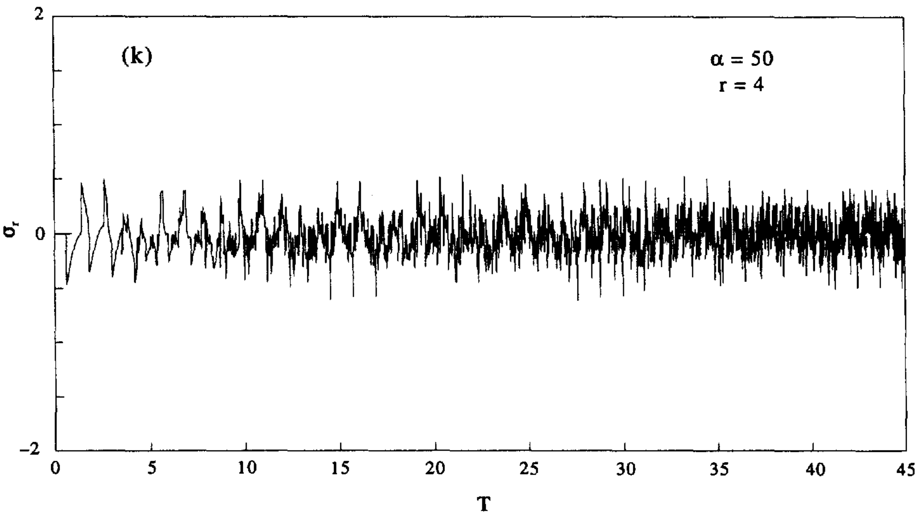


Fig. 6—*Continued*





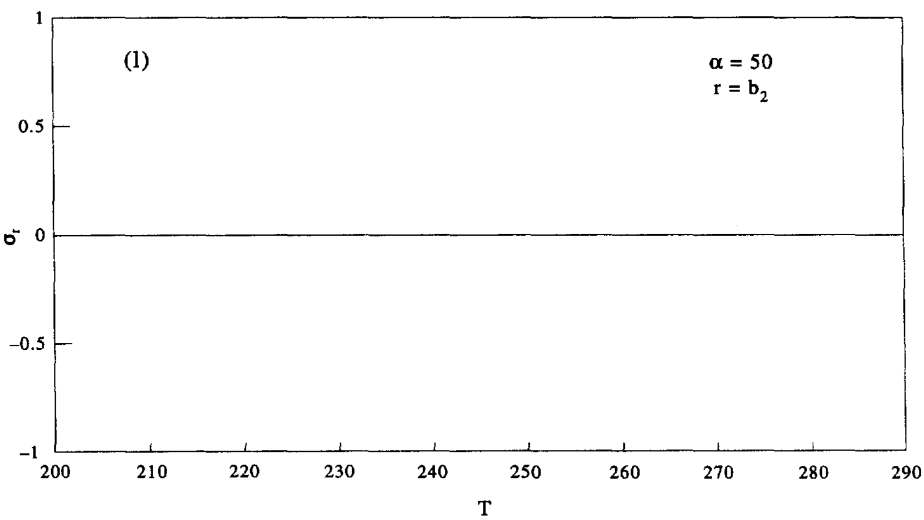


Fig. 6—*Continued*

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